

ON THICKNESS AND THINNESS OF BANACH SPACES

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ABSTRACT. The aim of this note is to complement and extend some recent results on Whitley's indices of thinness and thickness in three main directions. Firstly, we investigate both the indices when forming ℓ_p -sums of Banach spaces, and obtain formulas which show that they behave rather differently. Secondly, we consider the relation of the indices of the space and a subspace. Finally, every Banach space X containing a copy of c_0 can be equivalently renormed so that in the new norm c_0 is an M-ideal in X and both the thickness and thinness index of X equal 1.

1. INTRODUCTION

Let X be a Banach space, B_X its unit ball and S_X its unit sphere. Also, denote by $B(x, r)$ the closed ball with center in x and radius r . Whitley introduced in [22] the *index of thickness*,

$$T_W(X) = \inf \left\{ r > 0 : \exists (x_i)_{i=1}^n \subset S_X \text{ with } S_X \subset \bigcup_{i=1}^n B(x_i, r) \right\},$$

and the *index of thinness*,

$$t(X) = \inf \left\{ r > 0 : \forall (x_i)_{i=1}^n \subset S_X, \varepsilon > 0, \exists x \in S_X \text{ with } \max_i \|x_i - x\| < r + \varepsilon \right\}.$$

The subscript W in $T_W(X)$ is to indicate that this is Whitley's original definition. As is easily observed, if $\dim X < \infty$, $T_W(X) = 0$ and $t(X) = 2$ while if $\dim X = \infty$, $T_W(X), t(X) \in [1, 2]$. More difficult is the fact, proved by Whitley, that

$$T_W(\ell_p) = 2^{1/p} = t(\ell_p), \quad 1 \leq p < \infty.$$

Together with Whitley's observations that $T_W(c_0) = 1 = T_W(\ell_\infty)$, $t(c_0) = 1$ and $t(\ell_\infty) = 2$ it is clear that the whole range $[1, 2]$ of indices is possible and that $(1, 2)$ is covered by indices of reflexive spaces. We will see that, by choosing appropriate reflexive spaces X and Y , we may have $T_W(X) = 1$ and $t(Y) = 2$, but never $T_W(X) = 2$ nor $t(X) = 1$.

Whitley [22, Lemmas 3 and 8] also showed that $t(L_\infty[0, 1]) = T_W(L_\infty[0, 1]) = 2$. Recently (see [5] and [8]) it was shown that $T_W(L_p[0, 1]) = 2^{1/p}$ for $1 \leq p < \infty$. In [6, Example 3.6] it was shown that $t(L_1[0, 1]) = 2$ and in [20,

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Theorem 6.3] that $t(L_p[0, 1]) = 2^{1/p}$ for $p \geq 2$. In fact, it is clear from the proof of [20, Theorem 6.3] that $t(L_p[0, 1]) \leq 2^{1/p}$ for all $1 \leq p < \infty$. Rainis Haller (private communication) pointed out to us that for all $f \in S_{L_p}$ $\|f_i - f\|^p$ is almost 2 when $f_i = n^{1/p} \chi_{[i/n, (i+1)/n]}$. This shows that the lower bound is also $2^{1/p}$, hence $t(L_p[0, 1]) = 2^{1/p}$ for all $1 \leq p < \infty$.

Before proceeding, let us just mention that in [8] it is noted that when $\dim X = \infty$ and $S_X \subset \bigcup_{i=1}^n B(x_i, r)$, $(x_i)_{i=1}^n \subset S_X$, then $B_X \subset \bigcup_{i=1}^n B(x_i, r)$. Thus, for $\dim X = \infty$, the index

$$T(X) = \inf \left\{ r > 0 : \exists (x_i)_{i=1}^n \subset S_X \text{ with } B_X \subset \bigcup_{i=1}^n B(x_i, r) \right\}$$

equals $T_W(X)$. Note that when $\dim X < \infty$ we always have $T(X) = 1$ (while $T_W(X) = 0$). In this note we are only interested in calculating the index for infinite-dimensional Banach spaces and will thus take the freedom to use $T(X)$ in what follows to denote also $T_W(X)$.

Most of what is known concerning $T(X)$ and $t(X)$ can be found by combining [22], [5] and [8] (note that the two latter overlap a bit on T -results). The particular case when X is separable and $T(X) = 2$ is thoroughly described in terms of the almost Daugavet property in [17] and [19]. For the non-separable case see [14].

Yost introduced in [23] two indices

$$\mu_1(X) = \sup_{\substack{x_1, \dots, x_n \in S_X \\ n \in \mathbb{N}}} \inf_{x \in S_X} \frac{1}{n} \sum_{i=1}^n \|x_i - x\|$$

and

$$\mu_2(X) = \inf_{\substack{x_1, \dots, x_n \in S_X \\ n \in \mathbb{N}}} \sup_{x \in S_X} \frac{1}{n} \sum_{i=1}^n \|x_i - x\|.$$

He showed that we always have $\mu_1(X) \leq \mu_2(X)$ for any Banach space X .

Note that we can give similar formulations to the thinness and thickness index

$$t(X) = \sup_{\substack{x_1, \dots, x_n \in S_X \\ n \in \mathbb{N}}} \inf_{x \in S_X} \max_{1 \leq i \leq n} \|x_i - x\|$$

and

$$T(X) = \inf_{\substack{x_1, \dots, x_n \in S_X \\ n \in \mathbb{N}}} \sup_{x \in S_X} \min_{1 \leq i \leq n} \|x_i - x\|.$$

From these definitions we observe that $1 \leq \mu_1(X) \leq t(X) \leq 2$ and $1 \leq T(X) \leq \mu_2(X) \leq 2$ for any Banach space X .

The following example by Papini (see [21, Example 2]) shows that we may have $\mu_1(X) < t(X)$ and $T(X) < \mu_2(X)$.

EXAMPLE 1. Let $K = \{c\} \cup [a, b]$ with $c \notin [a, b]$ and consider $X = C(K)$. Then we have that $\mu_1(X) = \mu_2(X) = 3/2$ while $T(X) = 1$ and $t(X) = 2$.

It turns out that $T(X) = 2$ is equivalent to $\mu_2(X) = 2$ (see [21, Theorem 2.1]). It is clear that if $t(X) = 1$ then $\mu_1(X) = 1$, but the reverse implication seems to be unknown.

QUESTION 1. Does $\mu_1(X) = 1$ imply $t(X) = 1$?

Let us now present our contribution to the present theory of indices of thickness and thinness.

We start with some preparatory observations that we will use throughout. According to [2], see Proposition 3.3, a Banach space X is *almost square* if for every finite subset $(x_i)_{i=1}^n \subset S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| \leq 1 + \varepsilon$ for $i = 1, 2, \dots, n$. From [15], see Proposition 2.4, X is *octahedral* if for every finite subset $(x_i)_{i=1}^n \subset S_X$ and every $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| > 2 - \varepsilon$ for every $i = 1, 2, \dots, n$. The definition of octahedrality goes back to [11, p. 12]; we have not used the original definitions in order to have comparable formulations of almost squareness and octahedrality. Two basic observations are that $t(X) = 1$ is equivalent to X being almost square and that $T(X) = 2$ is the same as X being octahedral. We also prove two lemmas on thickness and thinness of c_0 -sums of Banach spaces and continue with a result that can be informally expressed as “thin spaces can have whatever thickness you like”. More precisely, we prove that for every $\alpha \in [1, 2]$, there is a Banach space X with $T(X) = \alpha$ while $t(X) = 1$. Then we will see that there are infinite-dimensional reflexive spaces X and Y for which $T(X) = 1$ and $t(Y) = 2$. We end Section 2 by studying the behaviour of thinness index when forming ℓ_p -sums of Banach spaces (see Propositions 2.6 and 2.8).

Next we address the problem of the relation between the thickness and thinness indices of the space and a subspace. Our question is motivated by the easy observation that $T(X) \geq T(X^{**})$ for any Banach space X . To generalize this observation we will need some concepts. First, recall the definition of an ideal (or locally 1-complemented subspace):

Definition 1.1. Let X be a Banach space and Y a subspace. Y is called an *ideal* in X if for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset X$ there exists $T : E \rightarrow Y$ such that

- (i) $Te = e$ for all $e \in Y \cap E$.
- (ii) $\|Te\| \leq (1 + \varepsilon)\|e\|$ for all $e \in E$.

If we instead of (ii) above have the stronger condition

$$(ii') \quad (1 + \varepsilon)^{-1}\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\| \text{ for all } e \in E,$$

then the ideal is called an *almost isometric ideal* (ai-ideal). So, in other words, an ai-ideal is an ideal (a locally 1-complemented subspace) where the local projections can be taken as almost isometries. The notion of an ai-ideal was defined and studied in [3].

With every ideal $Y \subset X$ there is an associated ideal projection $P : X^* \rightarrow X^*$ with $\|P\| = 1$ and $\ker P = Y^\perp$. It is observed in [3, Proposition 2.1] that the local projections can be taken as almost isometries whenever PX^* is a 1-norming subspace of X^* , that is, when the ideal is *strict*. There are, however, ai-ideals which are not strict, see [3, Example 1] or Remark 3.2 below.

Any Banach space X is an ai-ideal in X^{**} ; this fact is usually referred to as the principle of local reflexivity. Our generalization of the observation $T(X) \geq T(X^{**})$ is that ai-ideals in X always have at least as high thickness as the space itself. Also these subspaces always have lower or equal thinness index.

It is well-known that for a Banach space X we have $X^* = L_1(\mu)$ for some measure μ , i.e. X is *Lindenstrauss*, if and only if X is an ideal in every superspace. Some spaces are even ai-ideals in every superspace; these spaces are the Gurarii-spaces (see [3, Theorem 4.3]). Being an ai-ideal in every superspace will imply that any Gurarii-space has thickness index 2 and thinness index 1.

For our last and perhaps most interesting result, recall the definition of an M-ideal: A subspace Y of X is called an *M-ideal* if the associated ideal projection $P : X^* \rightarrow X^*$ is an L-projection, that is,

$$\|x^*\| = \|Px^*\| + \|(I - P)x^*\| \text{ for all } x^* \in X^*.$$

Proposition II.2.10 in [16] tells us: If $Y \subset X$ is a subspace isometric to c_0 , the original norm on Y can be extended to an equivalent norm on X in such a way that Y becomes an M-ideal in X with the new norm.

We will prove that any Banach space X containing an isomorphic copy of c_0 can be equivalently renormed so that, in this new norm, c_0 becomes an M-ideal in X and, moreover, both the thickness and thinness index of X equal 1.

Remark 1.1. That X can be equivalently renormed to have $T(X) = 2$ if and only if X contains an isomorphic copy of ℓ_1 was proved in [12, Theorem 9.2]. See also [14] or [21].

Sometimes we will say that X is thick, meaning $T(X) = 2$. Correspondingly, we sometimes say thin, meaning $t(X) = 1$. But, as can be seen e.g. from Proposition 2.4 or Proposition 3.2 below, Banach spaces may very well be both thick and thin at the same time, so the terms should not be read too literally.

2. SOME PREPARATORY OBSERVATIONS AND THIN SPACES WITH ANY KIND OF THICKNESS

One can easily observe that a Banach space X satisfies the condition $t(X) = 1$ if and only if it is almost square (see [2, Proposition 3.3]). Almost square spaces were introduced and studied in [2]. In [2] it is proved that if X is almost square, then every convex combination of slices of B_X has diameter 2. It is more or less folklore (see [11, p. 12]) that this slice property is in turn equivalent to X^* being octahedral. In [12] it is proved that octahedrality is equivalent to the condition that whenever the unit ball is covered by a finite number of balls, one of those balls already contains the unit ball itself, hence the thickness is 2. Thus we have

Proposition 2.1. *If $t(X) = 1$, then $T(X^*) = 2$.*

Remark 2.1. The converse implication, $T(X^*) = 2 \Rightarrow t(X) = 1$, is not true. As an example, take $X = C[0, 1]$. Then $T(X^*) = 2$ since X^* is octahedral, and also $t(X) = 2$ by [22, Lemma 8].

We now study thinness and thickness indices of c_0 -sums of sequences of Banach spaces. Let us first observe that c_0 -sums are always thin.

Lemma 2.2. *If (X_n) is a sequence of Banach spaces, then $t(c_0(X_n)) = 1$.*

Proof. Let $(x_i)_{i=1}^n \in S_{c_0(X_n)}$ and $\varepsilon > 0$. Find N such that $\|x_i(n)\| < \varepsilon$ for all $n \geq N$. Choose any $y_N \in S_{X_N}$ and define $y = (0, \dots, y_N, 0, \dots) \in S_{c_0(X_n)}$. Then $\max_i \|x_i - y\| < 1 + \varepsilon$, and so the lemma is proved. \square

Remark 2.2. Note that there appears to be a misprint in [5, Lemma 4.1]. The authors state that $t(\ell_\infty(X_n)) = 1$, but with $X_n = \mathbb{R}$, we have $t(\ell_\infty) = 2$ (see [22, Lemma 8]).

Remark 2.3. The thinness of a subspace may be strictly bigger than the thinness of the space itself. Indeed, $t(\ell_1) = 2$, but $t(c_0(\ell_1)) = 1$ by Lemma 2.2.

The corresponding result on the thickness index is harder. The proof is essentially that of [8, Theorem 2 (3)] and is omitted.

Lemma 2.3. *If (X_n) is a sequence of Banach spaces then $T(c_0(X_n)) = \inf_n T(X_n)$.*

Remark 2.4. Observe that Lemma 2.3 implies that there is in fact equality in [5, Proposition 2.14 (1)].

With the observations we have made so far at hand, we get the following result:

Proposition 2.4. *For every $\alpha \in [1, 2]$ there is a Banach space X with $T(X) = \alpha$ while $t(X) = 1$ and $T(X^*) = 2$.*

Proof. The statement “and $T(X^*) = 2$ ” is Proposition 2.1. From Whitley’s paper ([22, Lemma 4]) we know that $T(\ell_p) = 2^{1/p}$ for $1 \leq p < \infty$. From Lemma 2.3 we get that also $T(c_0(\ell_p)) = 2^{1/p}$. From Lemma 2.2 we know that $t(c_0(\ell_p)) = 1$. Thus the result has been proved for all $\alpha \in (1, 2]$. For $\alpha = 1$ consider $X = c_0$. \square

It is clear that $t(X) > 1$ and $T(X) < 2$ for all reflexive Banach spaces, this follows from e.g. Remark 1.1 and Proposition 2.1 above. The next proposition shows that all other possible values of $t(X)$ and $T(X)$ are covered by infinite-dimensional reflexive spaces.

Proposition 2.5. *For every $\alpha \in [1, 2)$ there is an infinite-dimensional reflexive Banach space X with $T(X) = \alpha$, and for every $\alpha \in (1, 2]$ there is an infinite-dimensional reflexive Banach space X with $t(X) = \alpha$.*

Proof. As we noted in the introduction, Whitley showed that $T(\ell_p) = 2^{1/p} = t(\ell_p)$ for $1 < p < \infty$ and this covers the interval $(1, 2)$.

Let Y be any infinite-dimensional reflexive Banach space. If we let $X = Y \oplus_\infty \mathbb{R}$, then it follows easily from (the proof of) [8, Lemma 3] that $T(X) = 1$. On the other hand if we let $X = Y \oplus_1 \mathbb{R}$ then $t(X) = 2$ by Corollary 2.7 below, since $t(\mathbb{R}) = 2$. \square

For ℓ_p -sums we have the following result:

Proposition 2.6. *Let Y and Z be Banach spaces and let $1 \leq p < \infty$. Then $X = Y \oplus_p Z$ satisfies $t(X) \geq ((t(Y) - 1)^p + 1)^{1/p}$.*

Proof. In [2, Lemma 5.6 and the preceding remark] it is noted that for $1 \leq p < \infty$, $X \oplus_p Y$ is never almost square, i.e. $t(X \oplus_p Y) > 1$. Since $((t(Y) - 1)^p + 1)^{1/p} = 1$ when $t(Y) = 1$ we may assume that $t(Y) > 1$.

For any $1 < \alpha < t(Y)$ there exist $(y_i)_{i=1}^n \subset S_Y$ and $\varepsilon > 0$ such that $\max_i \|y - y_i\| \geq \alpha + \varepsilon$ for all $y \in S_Y$. For a given $\alpha < t(Y)$ consider $(y_i)_{i=1}^n \subset S_Y$ and $\varepsilon > 0$ as above and define $x_i = (y_i, 0) \in S_X$ for $i = 1, 2, \dots, n$.

For $x = (y, z) \in S_X$ we have $\|x\|^p = \|y\|^p + \|z\|^p = 1$. We will also need that

$$1 - \|y\| = \frac{(1 - \|y\|)}{\|y\|} \|y\| = \left\| \frac{y}{\|y\|} - y \right\|.$$

By the triangle inequality and monotonicity of the ℓ_p -norms

$$\begin{aligned} \max_i \|x_i - x\|^p &= \max_i \|y_i - y\|^p + \|z\|^p \\ &\geq \left(\max_i \left\| y_i - \frac{y}{\|y\|} \right\| - \left\| \frac{y}{\|y\|} - y \right\| \right)^p + \|z\|^p \\ &\geq (\alpha + \varepsilon - 1 + \|y\|)^p + \|z\|^p \\ &\geq (\alpha + \varepsilon - 1)^p + \|y\|^p + \|z\|^p = (\alpha + \varepsilon - 1)^p + 1. \end{aligned}$$

Since $\alpha < t(Y)$ and $x \in S_X$ are arbitrary, we obtain $t(X) \geq ((t(Y) - 1)^p + 1)^{1/p}$. \square

Remark 2.5. As a general lower bound this is best possible since $t(\ell_p \oplus_p X) = 2^{1/p}$ by [5, Proposition 4.3] for any space with $t(X) = 2$, for example $X = \ell_1$.

Corollary 2.7.

- (i) If X and Y are Banach spaces, then $t(X \oplus_1 Y) \geq \max\{t(X), t(Y)\}$.
- (ii) If $(X_j)_{j=1}^\infty$ is a sequence of non-trivial Banach spaces, then $t(\ell_p(X_j)) \geq \sup_j ((t(X_j) - 1)^p + 1)^{1/p}$. Moreover, if $\sup_j t(X_j) = 2$, then $t(\ell_p(X_j)) = 2^{1/p}$.

Proof. It is clear that (i) holds. For the moreover part in (ii) it suffices to observe that the upper bound is proved in [5, Lemma 4.1]. \square

Proposition 2.8. Let X and Y be a Banach spaces. Then $t(X \oplus_\infty Y) = \min\{t(X), t(Y)\}$.

Proof. Let α and β be such that $\alpha < t(X)$ and $\beta < t(Y)$. Then there exist $(x_i)_{i=1}^n \subset S_X$, $(y_j)_{j=1}^k \subset S_Y$ and $\varepsilon > 0$ such that $\max_i \|x_i - x\| \geq \alpha + \varepsilon$ for all $x \in S_X$ and $\max_j \|y_j - y\| \geq \beta + \varepsilon$ for all $y \in S_Y$.

Without loss of generality we may assume that $k = n$ by just repeating some vectors. Define $z_i = (x_i, y_i)$, $1 \leq i \leq n$. Let $z = (x, y) \in X \oplus_\infty Y$ with $\|z\| = 1$. Then either $\|x\| = 1$ or $\|y\| = 1$ and hence

$$\max_i \|z_i - z\| = \max_i \{\|x_i - x\|, \|y_i - y\|\} \geq \min\{\alpha, \beta\} + \varepsilon.$$

Thus $t(X \oplus_\infty Y) \geq \min\{t(X), t(Y)\}$.

Note that the following holds in every Banach space: If two elements x' and x have norm one and $\|x' - x\| < a$ where $a \geq 1$, then for all $0 \leq r \leq 1$ we have $\|rx' - x\| < a$. Indeed,

$$\|rx' - x\| = \|rx' - rx + rx - x\| \leq r\|x' - x\| + 1 - r < a.$$

Now, suppose $\min(t(X), t(Y)) = t(X)$ and let $\varepsilon > 0$. Let $(x_i, y_i)_{i=1}^n$ be a finite set in the unit sphere of $X \oplus_\infty Y$. Let $u_i = x_i / \|x_i\|$ if $x_i \neq 0$. Then there is an element $x \in S_X$ such that $\max_i \|u_i - x\| < t(X) + \varepsilon$. Consider the

element $(x, 0)$ from the unit sphere of $X \oplus_\infty Y$. By the previous paragraph we get

$$\max_i \|(x_i, y_i) - (x, 0)\| = \max_i \{\|x_i\|u_i - x\|, \|y_i\|\} < t(X) + \varepsilon.$$

Finally, if $x_i = 0$ for every i , then for any $x \in S_X$ we have

$$\|(0, y_i) - (x, 0)\| = 1 \leq t(X).$$

□

The aim of the following example is to show that although $T(\ell_1) = t(\ell_1) = 2$, then by forming ℓ_p -sums these indices behave quite differently. In [8, Lemma 2] it was shown that $T(\ell_1 \oplus_2 \ell_1) = \sqrt{2 + \sqrt{2}}$, but $t(\ell_1 \oplus_2 \ell_1) = 2$ as we will see from the next proposition.

Proposition 2.9. $t(\ell_1 \oplus_p \ell_1) = 2$, $1 \leq p \leq \infty$.

Proof. If $p = 1$ or $p = \infty$ then the statement follows from Corollary 2.7 or Proposition 2.8, respectively, since $t(\ell_1) = 2$. Suppose now that $1 < p < \infty$.

Let $a^p + b^p = 1$. Then

$$\|(\pm ae_1, \pm be_1)\|^p = a^p + b^p = 1.$$

We can parametrize by $a = (\cos \theta)^{2/p}$ and $b = (\sin \theta)^{2/p}$. For any $x \in \ell_1$ and $a \in [0, 1]$ we have

$$\max_{\pm} \|\pm ae_1 - x\| = \max_{\pm} |a - x_1| + \sum_{n=2}^{\infty} |x_n| = a + |x_1| + \sum_{n=2}^{\infty} |x_n| = a + \|x\|.$$

Hence for $z = (x, y) \in S_Z$ we have $\|y\|^p = 1 - \|x\|^p$ and

$$\begin{aligned} (2.1) \quad \max_{\pm} \|(\pm ae_1, \pm be_1) - (x, y)\|^p &= (a + \|x\|)^p + \left(b + (1 - \|x\|^p)^{1/p}\right)^p \\ &= \left((\cos \theta)^{2/p} + \|x\|\right)^p + \left((\sin \theta)^{2/p} + (1 - \|x\|^p)^{1/p}\right)^p. \end{aligned}$$

Consider the continuous function on $[0, \pi/2] \times [0, 1]$ defined by

$$f(\theta, \xi) = \left((\cos \theta)^{2/p} + \xi\right)^p + \left((\sin \theta)^{2/p} + (1 - \xi^p)^{1/p}\right)^p.$$

For $\theta = \arccos(\xi^{p/2})$ we have

$$f(\arccos(\xi^{p/2}), \xi) = (2\xi)^p + (2(1 - \xi^p)^{1/p})^p = 2^p.$$

Let $\varepsilon > 0$. Since $f(\theta, \xi)$ is uniformly continuous there exists $\delta > 0$ such that

$$\|(\theta, \xi) - (t, u)\| < \delta \Rightarrow |f(\theta, \xi) - f(t, u)| < \varepsilon.$$

Choose a δ -net $(\xi_i)_{i=1}^n$ for the interval $[0, 1]$ and define $\theta_i = \arccos(\xi_i^{p/2})$.

Let $a_i = (\cos \theta_i)^{2/p}$ and $b_i = (\sin \theta_i)^{2/p}$ then for $1 \leq i \leq 4n$ and $1 \leq k \leq n$ define $z_i = (\pm a_k e_1, \pm b_k e_1)$.

For any $z = (x, y) \in S_Z$ we have by (2.1)

$$\max_i \|z_i - z\|^p = \max_i f(\theta_i, \|x\|).$$

Choose ξ_j such that $|\xi_j - \|x\|| < \delta$. Then

$$\max_i \|z_i - z\|^p = \max_i f(\theta_i, \|x\|) \geq f(\theta_j, \|x\|) \geq f(\theta_j, \xi_j) - \varepsilon = 2^p - \varepsilon.$$

This shows that $t(\ell_1 \oplus_p \ell_1) \geq \sqrt[p]{2^p - \varepsilon}$. Since $\varepsilon > 0$ was arbitrary we must have $t(\ell_1 \oplus_p \ell_1) = 2$. \square

3. THINNESS AND THICKNESS OF ALMOST ISOMETRIC IDEALS

Using Goldstine's theorem it is easy to see that $T(X^{**}) \leq T(X)$. This inequality may be strict. As an example $T(C[0, 1]) = 2$ while $T(C[0, 1]^{**}) = 1$ by [22, Lemma 3] since $C[0, 1]^{**}$ is not octahedral and can be viewed as a $C(K)$ space (cf. e.g. [1, Theorems 4.3.7 and 4.3.8]). Note that this example answers a question in [7] whether we always have $T(X) = T(X^{**})$. We will now put these observations into a broader perspective. A Banach space X is always an ai-ideal in X^{**} and so the observation above that $T(X^{**}) \leq T(X)$ is a very particular case of the following proposition.

Proposition 3.1. *If Y is an ai-ideal in X , then $T(X) \leq T(Y)$ and $t(Y) \leq t(X)$.*

Proof. Assume that $(y_i)_{i=1}^n$ is an r -net for S_Y . Let $\varepsilon > 0$. Let $x \in S_X$ and $E = \text{span}((y_i), x)$. Find an ε -isometry $T : E \rightarrow Y$. Let $z = Tx / \|Tx\| \in S_Y$. Then $\|z - Tx\| \leq \varepsilon$ since $(1 + \varepsilon)^{-1} \leq \|Tx\| \leq 1 + \varepsilon$. Now find j such that $\|y_j - z\| \leq r$, then

$$\|y_j - x\| \leq (1 + \varepsilon)\|y_j - Tx\| \leq (1 + \varepsilon)(r + \varepsilon).$$

Since $T(X)$ is an infimum and $\varepsilon > 0$ is arbitrary, we get $T(X) \leq T(Y)$.

Next assume that $(y_i)_{i=1}^n \subset S_Y$. Let $\varepsilon > 0$. Find $x \in S_X$ such that $\max \|y_i - x\| < t(X) + \varepsilon$. Let $x \in S_X$ and $E = \text{span}((y_i), x)$. Find ε -isometry $T : E \rightarrow Y$. Let $z = Tx / \|Tx\| \in S_Y$. Then, as above, $\|z - Tx\| \leq \varepsilon$. Now

$$\|y_j - z\| \leq \|y_j - Tx\| + \varepsilon \leq (1 + \varepsilon)\|y_j - x\| + \varepsilon \leq (1 + \varepsilon)(t(X) + \varepsilon).$$

Since $t(X)$ is an infimum and $\varepsilon > 0$ is arbitrary we have shown that $t(Y) \leq t(X)$. \square

An ai-ideal may well have strictly less thinness than its super-space.

Remark 3.1. Note that $t(c_0) = 1$ while $t(\ell_\infty) = 2$, so we have that $t(X) < t(X^{**})$ for $X = c_0$.

Proposition 3.1 will turn out to provide us with a class of spaces which are both thick and thin at the same time, namely the Gurarii-spaces. The “up to date reference” for Gurarii-spaces is [10] and the definition of a Gurarii-space can be found there. We will, however, use the alternative description of Gurarii-spaces ([3, Theorem 4.3]): The Gurarii-spaces is exactly the class of Banach spaces with the property that they form an ai-ideal in every super-space. It is known that there is only one separable Gurarii-space (up to linear isometry), and that a Gurarii-space X is universal in the sense that it contains an isometric copy of all Banach spaces Y with $\text{dens}(Y) \leq \text{dens}(X)$ (see [10]). Gurarii constructed the first separable such space in 1966 in his seminal paper [13].

Proposition 3.2. *If X is a Gurarii-space, then $T(X) = 2$ and $t(X) = 1$.*

Proof. Let X be a Gurarii-space. Then, by [3, Theorem 4.3], X is an ai-ideal in any super-space. In particular, it is an ai-ideal in $Y = C(B_{X^*}, w^*)$ (which does not have isolated points). Thus, by [22, Lemma 3] and Proposition 3.1 above, $T(X) \geq T(Y) = 2$.

To see that $t(X) = 1$, just note that, by [3, Theorem 4.3], X is an ai-ideal in $c_0(X)$ (X is isometrically isomorphic to $(X, 0, 0, \dots)$), and the result follows from Lemma 2.2 and Proposition 3.1. \square

Remark 3.2. Note that the embedding $X \rightarrow c_0(X)$ in the proof of Proposition 3.2 also makes X a 1-complemented subspace of $c_0(X)$. Thus, Gurarii-spaces X , embedded this way in $c_0(X)$, provide examples of non-strict, ai-ideals (see also [2, Example 1] where a non-strict ai-ideal in c_0 is constructed).

We have already mentioned that Gurarii-spaces are Lindenstrauss spaces. Lindenstrauss proves in his famous memoir (see [18, Theorem 6.1]) that when $X^* = L_1(\mu)$ and B_X has extreme points, then $X = X_1$, where X_1 is a subspace of some $C(K)$ space that contains 1. Using the unit, we see that $t(X) = 2$ whenever X is an L_1 -predual with $\text{ext}(B_X) \neq \emptyset$. In particular we get:

Proposition 3.3. *If X is a Gurarii-space, then $\text{ext}(B_X) = \emptyset$.*

We have just argued that a Lindenstrauss space has thinness index 2 as soon as $\text{ext}(B_X) \neq \emptyset$.

Proposition 3.4. *For every $\alpha \in [1, 2]$, there is a Lindenstrauss space with $t(X) = \alpha$. For any Lindenstrauss space we have $T(X^*) = 2$ and $t(X^{**}) = 2$.*

Proof. $T(X^*) = 2$ because X^* is octahedral and $t(X^{**}) = 2$ because X^{**} is a Lindenstrauss space with $\text{ext}(B_X) \neq \emptyset$. We need to consider the case $\alpha \in (1, 2)$. For this, let $r > 1$ and $X_r = \{f \in C[0, 1] : f(0) = rf(1)\}$. Then the spaces X_r are all L_1 -preduals (see e.g., [16, p. 83]). We are going to show that $t(X_r) = 1 + \frac{1}{r}$. Note that for all $f \in B_{X_r}$ we have $|f(1)| \leq \frac{1}{r}$.

To see that $t(X_r) \geq 1 + \frac{1}{r}$ let $f_1(x) = (1 - x) + \frac{1}{r}x$ and $f_2(x) = -f_1(x)$. If $\|g\| = 1$, then there is a point x_0 where $|g(x_0)| = 1$. Without loss of generality assume $g(x_0) = -1$. Then $\frac{1}{r} + 1 \leq |f_1(x_0) - g(x_0)| \leq \max \|f_i - g\|$. Hence $t(X_r) \geq 1 + \frac{1}{r}$.

To see that $t(X_r) \leq 1 + \frac{1}{r}$ let $f_1, f_2, \dots, f_n \in S_{X_r}$ and $\varepsilon > 0$. Find an interval $(a, 1)$ where $|f_i(x)| < \frac{1}{r} + \varepsilon$. Now choose any $g \in S_X$ with support on $(a, 1)$. Then $\|f_i - g\| < 1 + \frac{1}{r} + \varepsilon$, hence $t(X) \leq 1 + \frac{1}{r}$. \square

4. M-IDEAL-RENORMING OF COPIES OF c_0

Recall that Y is an M-ideal in X if the ideal projection $P : X^* \rightarrow X^*$, with $\|P\| = 1$ and $\ker P = Y^\perp$, is an L-projection. If X is an M-ideal in X^{**} (like c_0 is in ℓ_∞), X is called an *M-embedded* space.

M-ideals play a very important role in Banach space theory; the main reference for the theory of M-ideals is [16]. When X is a Banach space and X contains an isometric copy of c_0 , the c_0 -norm can always be extended to all of X such that, in this new norm, c_0 is an M-ideal in X , see [16, Proposition II.2.10]. We will now prove that we can extend in such a way

that $T(X) = t(X) = 1$ in the new norm. The proof is based on an idea which appears in [2, Theorem 3.14], which in turn used ideas from [4, Lemma 2.3].

Theorem 4.1. *If X contains an isomorphic copy of c_0 , then X can be renormed so that, in this new norm, c_0 becomes an M -ideal in X and $T(X) = t(X) = 1$.*

Proof. First, [9, Lemma II.8.1], we can renorm X so that it contains an isometric copy of c_0 . Denote by $\|\cdot\|$ this new norm on X . Let

$$A = \{Y \subset X : c_0 \subset Y, Y \text{ separable}\},$$

and order A by inclusion, i.e., $Y_2 \leq Y_1$ if $Y_2 \subset Y_1$. For every $Y \in A$ there exists, by Sobczyk's theorem, a projection P_Y onto c_0 with norm 2 or less. Let P_Y be such a projection and for each $Y \in A$ and $x \in Y$ let

$$\|x\|_Y := \max\{\|P_Y(x)\|, \|x - P_Y(x)\|\}.$$

By letting $\|x\|_Y = 0$ for $x \notin Y$ we can consider $(\|x\|_Y)_{Y \in A}$ as a net in $\Pi_{x \in X}[0, 3\|x\|]$. By Tychonoff's theorem this net has a convergent subnet, still denoted $(\|x\|_Y)_{Y \in A}$, and we may define

$$|||x||| = \lim_Y \|x\|_Y.$$

It is straightforward to show that $|||\cdot|||$ is a norm on X which satisfies $\frac{1}{2}\|x\| \leq |||x||| \leq 3\|x\|$. Also $|||\cdot|||$ extends the max norm $\|\cdot\|$ on c_0 . It was shown in [2, Theorem 3.14] that this norm is almost square, i.e., $t(X) = 1$ in this norm.

We want to show that c_0 is an M -ideal in X in this new norm. Let $x \in B_{(X, |||\cdot|||)}$, $y_1, y_2, y_3 \in B_{c_0}$ and $\varepsilon > 0$. Let $y_0 = 0$.

Let (z_n) be a sequence which is dense in c_0 and let $z_0 = 0$. Let $(\varepsilon_n)_{n=1}^\infty$ be a strictly decreasing null sequence of positive reals.

Let $Y_0 = \text{span}\{x, c_0\}$ and choose $Y_1 \in A$ with $Y_1 \supset Y_0$ such that

$$| |||x + y_i - z_0||| - \|x + y_i - z_0\|_{Y_1} | < \varepsilon_1$$

for $i = 0, 1, 2, 3$. Then for $n \geq 1$ inductively choose $Y_{n+1} \in A$ with $Y_{n+1} \supset Y_n$ such that

$$| |||x + y_i - z_k||| - \|x + y_i - z_k\|_{Y_n} | < \varepsilon_n$$

for every $k \leq n$ and $i = 0, 1, 2, 3$. (Note that the inequality above holds also for every $Y \in A$ with $Y \supset Y_n$.) Put $Y = \overline{\cup_{n=1}^\infty Y_n}$. Note that $Y \in A$ as $c_0 \subset Y$ and Y is separable. Observe that for $i = 0, 1, 2, 3$ and all $n \geq k$ we have

$$\begin{aligned} & | |||x + y_i - z_k||| - \|x + y_i - z_k\|_Y | \\ & \leq | |||x + y_i - z_k||| - \|x + y_i - z_k\|_{Y_n} | < \varepsilon_n, \end{aligned}$$

so $|||x + y_i - z_k||| = \|x + y_i - z_k\|_Y$ as $\varepsilon_n \downarrow 0$. In particular, we have

$$\|x - P_Y(x)\| \leq \|x\|_Y = \|x + y_0 - z_0\|_Y = |||x + y_0 - z_0||| \leq 1.$$

Let $z = P_Y(x)$. Choose j such that $\|z - z_j\|_{c_0} = |||z - z_j||| < \varepsilon$. Then we have

$$|||x + y_i - z||| \leq |||x + y_i - z_j||| + \varepsilon = \|x + y_i - z_j\|_Y + \varepsilon,$$

$$\begin{aligned}\|x + y_i - z_j\|_Y &= \max\{\|P_Y(x) + y_i - z_j\|, \|x - P_Y(x)\|\} \\ &\leq \max\{\|y_i\| + \|z - z_j\|, 1\} \leq 1 + \varepsilon,\end{aligned}$$

To see that $T(X) = 1$ we will show that in the new norm $B(e_1, 1) \cup B(-e_1, 1)$ covers $B_{(X, \|\cdot\|)}$.

$$|||x + z||| = \|x + z\|_Y$$

We get

$$\begin{aligned} ||x \pm e_1|| &= \|x \pm e_1\|_Y = \max(\|P_Y(x \pm e_1)\|, \|x \pm e_1 - P_Y(x \pm e_1)\|) \\ &= \max(\|P_Y(x) \pm e_1\|, \|x - P_Y(x)\|) \leq \max(\|P_Y(x) \pm e_1\|, 1) \end{aligned}$$

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